The Natural Logarithm

Worksheet Solutions

1. For each of the following equations, rearrange to make x the subject.

   (a) $y = 3^{2x}$

   (b) $p = e^{4x-8} + 6$

   (c) $N = Ae^{-kx}$

   (d) $q = 2^x \times 4^{2x}$

a) $y = 3^{2x}$

$\Rightarrow \log_3 y = \log_3 3^{2x}$

$\Rightarrow \log_3 y = 2x$

$\Rightarrow x = \frac{1}{2} \log_3 y.$

Alternately $\ln y = \ln 3^{2x}$

$\Rightarrow \ln y = 2x \ln 3$

$\Rightarrow 2x = \frac{\ln y}{\ln 3}$

$\Rightarrow x = \frac{1}{2} \frac{\ln y}{\ln 3}.$

b) $p = e^{4x-8} + 6$

$\Rightarrow p - 6 = e^{4x-8}$

$\Rightarrow \ln(p-6) = 4x - 8$

$\Rightarrow x = \frac{\ln(p-6) + 8}{4}.$

c) $N = Ae^{-kx}$

$\Rightarrow \ln N = \ln(Ae^{-kx})$

$\Rightarrow \ln N = \ln A + \ln e^{-kx}$

$\Rightarrow \ln N = \ln A - kx$

$\Rightarrow kx = \ln N - \ln A$

$\Rightarrow x = \frac{1}{k} \ln \frac{N}{A}$.

d) $q = 2^x \times 4^{2x} = 2^x \times ((2)^2)^{2x}$

$\Rightarrow q = 2^x \times 2^{4x}$

$\Rightarrow q = 2^{5x}$

$\Rightarrow \log_2 q = \log_2 2^{5x}$

$\Rightarrow \log_2 q = 5x$

$\Rightarrow x = \frac{1}{5} \log_2 q$.

Alternately: $\ln q = \ln 2^{5x}$

$\Rightarrow \ln q = 5x \ln 2$

$\Rightarrow x = \frac{1}{5} \frac{\ln q}{\ln 2}$.

2. Find the following integrals:

a) $$\int \frac{1}{x}\ dx$$

b) $$\int (x^2 + \frac{1}{x})\ dx$$

c) $$\int (2 + \frac{1}{x^2} - \frac{1}{x})\ dx$$

d) $$\int (e^x + \frac{2}{x})\ dx$$

a) $\int \frac{1}{x}\ dx = \ln|x| + C.$

b) $\int x^2 + \frac{1}{x}\ dx = \frac{x^3}{3} + \ln|x| + C.$

c) $\int 2 + \frac{1}{x^2} - \frac{1}{x}\ dx = \int 2 + x^{-2} - \frac{1}{x}\ dx$

$= 2x - \frac{1}{x} - \ln|x| + C.$

d) $\int e^x + \frac{2}{x}\ dx$

$= e^x + 2\ln|x| + C$

$= e^x + \ln(x^2) + C.$

No modulus needed as $x^2 \geqslant 0$.

3. By using the fact that for a differentiable function $f(x)$

$$\int \frac{f'(x)}{f(x)}\ dx = \ln \mid f(x) \mid + C.$$

Find the following integrals:

a)
$$\int \frac{2x+2}{x^2+2x+66}\ dx$$

b)
$$\int \frac{3x^2+6x}{x^3+3x^2} + 4\ dx$$

c)
$$\int \frac{2e^{2x}}{e^{2x}-4}\ dx$$

d)
$$\int \frac{x^3+6x^2-4}{x^4+8x^3-16x+2}\ dx$$

a) $\displaystyle\int \frac{2x+2}{x^2+2x+66}\ dx$

$= \ln \left| x^2 + 2x + 66 \right| + C.$

b) $\displaystyle\int \frac{3x^2+6x}{x^3+3x^2} + 4\ dx$

$= \ln | x^3 + 3x^2 | + 4x + C.$

c) $\displaystyle\int \frac{2e^{2x}}{e^{2x}-4}\ dx$

$$= \ln \left| e^{2x} - 4 \right| + C.$$

d) $$\int \frac{x^3 + 6x^2 - 4}{x^4 + 8x^3 - 16x + 2} \, dx$$

$$\frac{d}{dx}\left(x^4 + 8x^3 - 16x + 2\right) = 4x^3 + 24x^2 - 16$$
$$= 4\left(x^3 + 6x^2 - 4\right).$$

$$\int \frac{x^3 + 6x^2 - 4}{x^4 + 8x^3 - 16x + 2} \, dx$$

$$= \frac{1}{4} \int \frac{4\left(x^3 + 6x^2 - 4\right)}{x^4 + 8x^3 - 16x + 2} \, dx$$

$$= \frac{1}{4} \ln \left| x^4 + 8x^3 - 16x + 2 \right| + C.$$

4. Calculate the following definite integrals:

a)
$$\int_1^2 \frac{1}{x}\ dx$$

b)
$$\int_3^4 \frac{1}{x^2} - \frac{1}{x}\ dx$$

c)
$$\int_2^4 \frac{3x^2+2x}{x^3+x^2}\ dx$$

d)
$$\int_{-4}^{-1} \frac{x^5}{x^6+4}\ dx$$

a) $\int_1^2 \frac{1}{x}\ dx = \Big[\ln|x|\Big]_1^2$

$= \ln|2| - \ln|1|$

$= \ln 2 - 0 = \ln 2.$

b) $\int_3^4 \frac{1}{x^2} - \frac{1}{x}\ dx = \int_3^4 x^{-2} - \frac{1}{x}\ dx$

$= \Big[-\frac{1}{x} - \ln|x|\Big]_3^4$

$= \left(-\frac{1}{4} - \ln 4\right) - \left(-\frac{1}{3} - \ln 3\right)$

$= -\frac{1}{4} - \ln 4 + \frac{1}{3} + \ln 3$

$= \ln\left(\frac{3}{4}\right) + \frac{1}{12}.$

c) $$\int_2^4 \frac{3x^2+2x}{x^3+x^2}\,dx$$

$$= \Big[\ln|x^3+x^2|\Big]_2^4$$

$$= \ln|4^3+4^2| - \ln|2^3+2^2|$$

$$= \ln|76| - \ln|12|$$

$$= \ln\left(\frac{76}{12}\right) = \ln\left(\frac{19}{3}\right).$$

d) $$\int_{-4}^{-1} \frac{x^5}{x^6+4}\,dx$$

$$= \frac{1}{6}\int_{-4}^{-1} \frac{6x^5}{x^6+4}\,dx$$

$$= \frac{1}{6}\Big[\ln|x^6+4|\Big]_{-4}^{-1}$$

$$= \frac{1}{6}\ln(5) - \frac{1}{6}\ln(4100) = \frac{1}{6}\ln\left(\frac{1}{820}\right).$$

5. Find the area bounded by the curve $y = \frac{4x-4}{x^2-2x+3}$ and the lines $x = 1$ and $x = 9$.

$$\text{Area} = \int_1^9 \frac{4x-4}{x^2-2x+3}\,dx$$

$$= 2\int_1^9 \frac{2x-2}{x^2-2x+3}\,dx$$

$$= 2\Big[\ln|x^2-2x+3|\Big]_1^9$$

$$= 2\ln|66| - 2\ln|2|$$

$$= 2\ln(33) \text{ units}^2.$$

6. Find the area bounded by the curve $y = \frac{\frac{1}{x}}{\ln x}$ and the lines $x = 3$ and $x = 4$.

$$\text{Area} = \int_3^4 \frac{\frac{1}{x}}{\ln x} \, dx$$

$$= \Big[\ln |\ln x| \Big]_3^4$$

$$= \ln|\ln 4| - \ln|\ln 3|$$

$$= \ln\left(\frac{\ln 4}{\ln 3}\right) \text{ units}^2.$$